FINAL: ALGEBRAIC GEOMETRY

Date: $28^{\textbf{th}}$ **April 2015**
The Total points is **110** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity**.

(1) (10 points) Let X be an affine algebraic set. Show that X is connected iff the only idempotents in its coordinate ring are 0 and 1.

(2) (5+15=20 points) Define reduced scheme. Let X be a reduced noetherian affine scheme. Show that there exist at least one and at most finitely points $x_1, \ldots, x_n$ in X such that the stalk(s) $\mathcal{O}_{X,x_i}$ are fields.

(3) (20 points) Let X be a scheme. Prove or Disprove
   (a) $X, \mathcal{O}_X$ is reduced iff there exist an affine open cover $\{U_i\}$ of X such that $(U_i, \mathcal{O}_X|_{U_i})$ is reduced for all i.
   (b) $X, \mathcal{O}_X$ is irreducible iff there exist an affine open cover $\{U_i\}$ of X such that $(U_i, \mathcal{O}_X|_{U_i})$ is irreducible for all i.

(4) (5+15=20 points) When is a morphism of schemes called separated? Let Y be a separated Z-scheme. Then for any Y-scheme X_1, X_2, show that the canonical morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$ is a closed immersion.

(5) (5+15=20 points) Let X be a k-scheme and $x \in X$ be a point. When is x called a nonsingular point of X? Let $f : X \to Y$ be a birational proper k-morphism of integral k-curves. Let U be an open subset of Y such that every point of U is nonsingular and $V = f^{-1}(U)$. Show that $f|_V$ is an isomorphism.

(6) (20 points) Let X be the projective plane curve over a field k given by the irreducible homogeneous polynomial $zy^2 - x^3$. Show that the normalization of X is isomorphic to $\mathbb{P}^1$.